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On the Manifolds of Total Collapse Orbits and of Completely Parabolic Orbits for the n-Body Problem*

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In this paper we show for most choices of masses in the coplanar n-body problem and for certain masses in the three-dimensional n-body problem that the set of initial conditions leading to complete collapse forms a smooth submanifold in phase space where the dimension depends upon properties of the limiting configuration. A similar statement holds for completely parabolic motion. By a proper scaling, these two motions become dual to each other in the sense that one forms the unstable set while the other forms the stable set of a particular set of points in the new phase space. Most of the paper is devoted to solving the Painleve–Wintner problem which asserts that these types of orbits cannot enter in an infinite spin.

1. INTRODUCTION

The purpose of this paper is to analyze total collapse and completely parabolic orbits of Newtonian n-body systems. In particular, we solve a problem posed by Painlevé and discussed by Wintner [13, p. 283] as to whether colliding particles can enter into an infinite spin. We show they cannot. As a corollary we improve upon a classical result due to Weierstrass by showing that if a system suffers a total collapse, then, for all time, the system admits *no* natural rotation! Furthermore, we show for a large class of n-body systems that the set of initial conditions leading to total collapse or to completely parabolic motion lies in the finite union of smooth submanifolds. Indeed, it turns out that these two types of motion are dual to

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each other in the sense that one can be viewed as determining the stable set while the other determines the unstable set for a particular set of orbits.

For collisions occurring at time $t=0$, it is known that the colliding particles approach their limiting point essentially like $t^{2/3}$. (For total collapse of the system, see Wintner [13]; for arbitrary collisions see Pollard and Saari [7], and Saari [9].) Wintner asked whether during this approach to collision the colliding particles could go into an infinite spin. More precisely, let $\mathbf{r}_i$ be the vector position of the ith particle relative to the center of mass of the system. Then for total collapse, $\mathbf{r}_i \to \mathbf{0}$ for all choices of i. Suppose $|\mathbf{r}_i - \mathbf{r}_j|/t^{2/3} \to A_{ij}$, as $t \to 0$, where i, j are the indices of colliding particles and A_{ij} is some non-negative constant. Wintner asked whether this implies that $(\mathbf{r}_i - \mathbf{r}_j)/t^{2/3}$ must approach a vector limit. Using properties peculiar to the three-body problem, Siegel [12] showed for the three-body problem that both the assumption and the conclusion hold for triple collisions. (It is easy to show that the result holds for binary collisions.)

We answer the Painleve–Wintner question in general by showing for all values of n and for *all* types of collisions that a completely collapsing system cannot enter into an infinite spin. Furthermore, with this result, we show for a large class of collapse problems that the set of initial conditions leading to this type of collision is a finite union of smooth submanifolds where the dimensions of the submanifolds depend upon the "type" of collision. (Siegel did this for the three-body problem.)

In the analysis of expanding gravitational systems, similar questions arise. In expanding systems there is one type of motion (which for the two-body problem corresponds to zero energy "escape velocity") where the distances between particles separate like $t^{2/3}$ as $t \to \infty$. As in the collision problem, for a large class of problems this expansion between particles is asymptotic to constant multiples of $t^{2/3}$. (For $n=3$, see Chazy [1] and Hulkower [2]; for $n \geqslant 3$, see Pollard [6], Saari [9], and Marchal and Saari [4].)

The same type of questions asked about collisions can be asked about parabolic motion [9]. Namely, if $|\mathbf{r}_i - \mathbf{r}_j|/t^{2/3}$ approaches a positive limit, does $(\mathbf{r}_i - \mathbf{r}_j)/t^{2/3}$ approach a vector limit? Hulkower showed that it does for the coplanar three-body problem. We show for general n-body systems with completely parabolic motion that the system cannot enter into an infinite spin of the type specified here. ("Higher order" spins, or a spin in physical space may be possible in one case.) As a corollary, we complete Hulkower's work by extending his statement to the three-dimensional three-body problem. Furthermore, we show for a large class of problems that the set of initial conditions leading to this type of expansion can be described in terms of a finite union of smooth submanifolds. (Hulkower did this for the coplanar three-body problem.)

The intent of both "spin" problems is to obtain refined asymptotic approximations for the respective motions. We shall adopt a slightly different interpretation. Let $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n)$. The asymptotic analysis can be viewed as approximating the behavior of $|\mathbf{r}|$, the radius of $\mathbf{r}$ in a spherical coordinate representation. By scaling the problem, $\mathbf{r}/t^{2/3}$ gives information describing the motion in S^{3n-1}, a sphere in R^{3n}. With this interpretation, the spin problem is to establish a limiting rotational position of the orbits in S^{3n-1}. We go beyond this by showing that collision motion and parabolic motion are dual to each other in the sense that they correspond, respectively, to the unstable set and the stable set of particular subsets of S^{3n-1}. In many cases these stable and unstable sets are smooth submanifolds, and it is in this fashion that we obtain our submanifold statements. Incidently, since we use the stable manifold theorem to establish these statements, we also obtain improved estimates on the asymptotic analysis of the orbits.

Precise statements of the results will be given in the following sections. In Sections 2 and 3 we show that total collapse and completely parabolic orbits cannot admit an infinite spin. In Section 4 we use the stable manifold theorem to obtain the manifold statements.

The notation is standard. Assume that the center of mass of the system is located at the origin of an inertial coordinate system. Let $\mathbf{r}_i$, $\mathbf{v}_i$, and m_i denote, respectively, the position vector, the velocity vector, and the mass of the ith particle. The same letters will denote the length of a vector, e.g., $r_i = |\mathbf{r}_i|$, $v_i = |\mathbf{v}_i|$, $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$.

Assuming the units are selected so that the gravitational constant equals unity, the equations of motion are

$$m_i \ddot{\mathbf{r}}_i = \frac{\partial U}{\partial \mathbf{r}_i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

where

$$U = \sum_{1 \leqslant i < j \leqslant n} m_i m_j r_{ij}^{-1}.$$

The integrals of motion are

$$T = U + h, \tag{1.2}$$

where

$$2T = \sum m_i v_i^2$$

and

$$\mathbf{c} = \sum m_i \mathbf{r}_i \times \mathbf{v}_i. \tag{1.3}$$

Constants h and $\mathbf{c}$ are determined by initial conditions.

These constants play a role in total collapse and completely parabolic motion in the following way. Weierstrass showed that a necessary condition for total collapse ($\mathbf{r}_i \to \mathbf{0}$ as $t \to 0$, $i = 1, 2, \ldots, n$) is $\mathbf{c} = \mathbf{0}$. For completely parabolic motion (as $t \to \infty$, $|\mathbf{r}_i - \mathbf{r}_j|$ is bounded above and below by constant multiples of $t^{2/3}$, $i \neq j$) the value of h must be zero. (See, for example, Pollard [6].)

A measure of the growth of the system is given by

$$2I = \sum_{i=1}^{n} m_i r_i^2 = M^{-1} \sum_{1 \leqslant i < j \leqslant n} m_i m_j r_{ij}^2,$$

where M is the total mass of the system. The two summations agree because the center of mass is located at the origin of the inertial coordinate system.

2. Stopping the Spin for Total Collapse and Completely Parabolic Motion

In this and the next section we discuss the possibility of particles entering into an infinite spin. Our analysis starts with a scaling of the variables.

Define $\mathbf{R}_i = \mathbf{r}_i / t^{2/3}$, $i = 1, 2, \ldots, n$.

The equations of motion for $\mathbf{R}_i$ are

$$\ddot{\mathbf{R}}_i t^{2/3} + \frac{4}{3} \dot{\mathbf{R}}_i t^{-1/3} - \frac{2}{9} \mathbf{R}_i t^{-4/3} = t^{-4/3} \sum_{\substack{j=1 \\ j \neq 1}}^{n} \frac{m_j (\mathbf{R}_j - \mathbf{R}_i)}{|\mathbf{R}_j - \mathbf{R}_i|^3}$$

$$(\text{def.}) = t^{-4/3} \frac{1}{m_i} \frac{\partial U(\mathbf{R})}{\partial \mathbf{R}_i}$$

or

$$\mathbf{R}_i'' t^2 + \frac{4}{3} \dot{\mathbf{R}}_i = \frac{2}{9} \mathbf{R}_i + \frac{1}{m_i} \frac{\partial U}{\partial \mathbf{R}_i}, \qquad i = 1, 2, \ldots, n.$$

This is an Euler system of differential equations, so the change of variables $t = e^u$ converts the system to

$$\ddot{\mathbf{R}}_i + \frac{1}{3} \mathbf{R}_i' = \frac{2}{9} \mathbf{R}_i + \frac{1}{m_i} \frac{\partial U}{\partial \mathbf{R}_i}, \tag{2.1}$$

where the prime denotes d/du.

As $u \to \infty$, $t \to \infty$; so $u \to \infty$ is the asymptotic limit for our study of completely parabolic motion. As $u \to -\infty$, $t \to 0$; so $u \to -\infty$ corresponds to the limit for total collapse. Since $\sum m_i \mathbf{r}_i = 0$, we have $\sum m_i \mathbf{R}_i = \mathbf{0}$. Assume that this condition holds for all systems of vectors used in this paper.

It is known both for the total collapse problem (Wintner [13]) and for completely parabolic motion (Saari [9]) that both $\mathbf{V}_i = \mathbf{R}_i'$ and $\mathbf{R}_i''$ approach zero as u approaches the appropriate limit. Thus the solutions for the two types of motion being studied must approach set

$$\mathrm{CC} = \left\{ (\mathbf{R}, \mathbf{V}) \mid \mathbf{R} = (\mathbf{R}_i, \ldots, \mathbf{R}_n), \frac{2}{9} \mathbf{R}_i + m_i^{-1} \frac{dU}{\partial \mathbf{R}_i} = \mathbf{0},\ i = 1, 2, \ldots, n; \right.$$

$$\left. \mathbf{V} \in (R^3)^n,\ \mathbf{V}_i = \mathbf{0},\ i = 1, 2, \ldots, n \right\}.$$

An element of set CC satisfying the condition imposed upon the position vectors is called a central configuration (Wintner [13, pp. 295–305]). (In the usual definition of a central configuration, the term 2/9 is replaced with an arbitrary scalar. This change only effects the scale of the resulting configuration in physical space.) Two central configurations in R^3 are identified whenever a rotation takes one onto the other. (It is easy to see that CC contains all possible rotations of any configuration from the set.) With this identification, there are only four central configurations for the three-body problem: the equilateral triangle and three collinear configurations where the distances between particles are determined by the values of the masses and their ordering along the line. For $n > 3$ it is unknown, but conjectured (Wintner) that there are only a finite number of central configurations. If this conjecture is false, then for coplanar configurations it is false for at most a lower-dimensional subset of masses (Palmore [5]). It is easy to show that in the general case the conjecture is true for an open set of masses.

Let $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_n)$ correspond to a central configuration and let $M_A = \{\Omega \mathbf{A}_1, \Omega \mathbf{A}_2, \ldots, \Omega \mathbf{A}_n) \mid \Omega \in SO(3)\} \times \mathbf{0}\}$. Set M_A corresponds to all possible rotations of configuration $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ in physical space R^3. Thus set CC is the union of sets M_A where $\mathbf{A}$ is a central configuration. It follows from the above stated results that, in general, CC is the finite union of such sets.

The importance of set CC is given in the following statement:

THEOREM 2.1. *The stable set of CC corresponds to completely parabolic orbits of the n-body problem. The unstable set of CC corresponds to total collapse orbits. $A \in CC$ corresponds to motion which leads to complete collapse as $t \to 0$, completely parabolic motion as $t \to \infty$, and motion which retains the same configuration for all time.*

Proof. $\mathbf{A} \in \mathrm{CC}$ is a rest point Eq. (2.1). The last sentence of the theorem follows immediately and it is a classical result (Wintner [13]).

One direction of the theorem follows from the asymptotic analysis showing that for the motion being discussed, the solutions of Eq. (2.1) must tend to set CC as u approaches the appropriate limit.

To prove the converse, we use the result (Saari [11]) that for any n, there exist positive constants A_1 and A_2 which serve as lower and upper bounds for the distance between any two component vectors in CC. Thus, if a solution of Eq. (2.1) approaches CC as $u \to -\infty$, then this corresponds to motion where $t^{-2/3} r_{ij}$, $i \neq j$, is bounded above as $t \to 0$. This is a sufficient condition for the singularity of the system to corresponds to a collision. (Pollard and Saari [8]). Trivially, this collision must correspond to a total collapse.

Correspondingly, if a solution of Eq. (2.1) tends to CC as $u \to \infty$, then $r_{ij}/t^{2/3}$, $i \neq j$, is bounded both above and away from zero as $t \to \infty$. This corresponds to completely parabolic motion. This completes the proof.

The obvious goal is to refine Theorem 2.1 so it would read that the stable (the unstable) set of CC can be expressed as the union of the stable (respectively, the unstable) sets of the points of CC. There are two problems standing in the way of such a conclusion. The first is the possibility of a continuum of central configurations. Should this be the case, then as an orbit approaches CC it may approach a continuum of configurations, but not one particular point! Fortunately, at least for the coplanar problem, this cannot happen in general because for most choices of the masses the central configurations are isolated.

However, even should the configurations be isolated, CC still contains smooth submanifolds: as we have shown, if $\mathbf{A} \in \text{CC}$, then so is M_A. Therefore, the possibility remains that an orbit approaching CC approaches M_A, yet it does not tend to any particular point on M_A. Notice that this is a restatement of the Painleve–Wintner spin problem. (Should an orbit approach M_A for some $\mathbf{A}$, then the hypothesis of their problem is trivially satisfied. This is because $R_{ij} = |\mathbf{r}_i - \mathbf{r}_j|/t^{2/3}$ approaches a fixed positive constant; namely, the length of the appropriate edge of the configuration.)

We analyze this problem by separating the motion into the part describing changes in the configurations and the part describing the rotation of the configuration (the $SO(3)$ action). Then, we show for all orbits approaching CC that the rotational part must approach a limit. This answers the Painleve–Wintner problem for *all* orbits, even if there should exist a continuum of central configurations!

THEOREM 2.2. *A total collapse orbit or a completely parabolic orbit approaches a limiting orientation as t approaches the appropriate limit.*

Since for most (and maybe for all) coplanar systems the central configurations are isolated, a refined version of Theorem 2.1 holds.

COROLLARY. *Let* $n = 2, 3$. *The stable (the unstable) set of CC corresponds to the union of the stable (respectively, the unstable) sets of the points of CC.*

COROLLARY. *Let $n > 3$. For the coplanar problem and for most choices of the masses, the stable (the unstable) set of CC corresponds to the union of the stable (respectively, the unstable) sets of the points of CC. The same conclusion holds in the general case for an open set of choices of the masses.*

As we stated earlier, Siegel proved Theorem 2.2 for the case $n = 3$ when $t \to 0$. Hulkower proved it for the coplanar three-body problem as $t \to \infty$. The approach used by them was to introduce a rotating coordinate system and then find the asumptotic behavior of the motion within this rotating system. By doing this, the motion in the rotating coordinate system provided estimates on the rate of spin, which in turn allowed them to show that infinite spin was impossible.

Such an approach does not seem to work for $n > 3$ because it requires more knowledge about central configurations than is currently available. For example, it requires the limiting configurations to be isolated. Furthermore, the choice of a rotating coordinate system is not obvious. We take a different approach by allowing the geometry of $(R^3)^n$ to determine the appropriate rotation.

Proof of theorem. For $\mathbf{y} = (\mathbf{y}_1, ..., \mathbf{y}_n)$, $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n) \in (R^3)^n$ define $\langle \mathbf{x}, \mathbf{y} \rangle = \sum^n m_i \mathbf{x}_i \cdot \mathbf{y}_i$. With this notation, the center-of-mass restriction imposed on the components of $\mathbf{R}$ is equivalent to requiring $\mathbf{R}$ to lie in the subspace $\{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{E}_i \rangle = 0$, where $\mathbf{E}_i = (\mathbf{e}_i, ..., \mathbf{e}_i)$, $i = 1, 2, 3$, and $\mathbf{e}_i \in R^3$ has unity in the ith component and zero in all others.$\}$ Denote this linear subspace by $R^{3(n-1)}$.

We shall express vectors in $R^{3(n-1)}$ in terms of spherical coordinates where $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ denotes the radius of vector $\mathbf{x}$. For $a > 0$, let S_a^{3n-4} be the sphere of radius a in $R^{3(n-1)}$. We provide a foliation for S_a^{3n-4} by describing the leaves in terms of orbits of a Lie group. Namely, if $\mathbf{x} \in S_a^{3n-4}$, let $M_{\mathbf{x}} = \{\Omega \mathbf{x} = (\Omega \mathbf{x}_1, ..., \Omega \mathbf{x}_n) \mid \Omega \in SO(3)\}$. Set (or leaf) $M_{\mathbf{x}}$ corresponds to all possible rotations in R^3 of the configuration defined by $\mathbf{x}$.

Let $\mathbf{v}$ be a vector field in $TR^{3(n-1)}$. We decompose $\mathbf{v}$ into three parts, $\mathbf{v} = \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3$, in the following manner. Vector $\mathbf{W}_3$ is the radial component; i.e., if $\mathbf{v}(\mathbf{x}) \in T_{\mathbf{x}} R^{3(n-1)}$, then, in the obvious fashion, identify $T_{\mathbf{x}} R^{3(n-1)}$ with $R^{3(n-1)}$. Define $\mathbf{W}_3(\mathbf{x})$ to be the projection of $\mathbf{v}$ on the ray $\lambda \mathbf{x}$. This vector describes the change in scale of the configuration. Vector $\mathbf{W}_1$ is the rotational component defined by projecting $\mathbf{v}(x)$ on the distribution of the foliation defined above. Vector $\mathbf{W}_2$ is what remains, and it gives the change in the configuration.

With this notation, Eq. (2.1) can be expressed as

$$\mathbf{R}' = \mathbf{v} = \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3, \tag{2.2}$$

$$\mathbf{v}' = \frac{-1}{3} \mathbf{v} + \nabla J,$$

where

$$J = \tfrac{2}{9}I(\mathbf{R}) + U(\mathbf{R}) = \tfrac{2}{9}\left(\tfrac{1}{2}\sum m_i\mathbf{R}_i^2\right) + U(\mathbf{R}).$$

To prove the theorem, we will show that either $|\mathbf{W}_1|$ is identically zero or it approaches zero exponentially fast. Two proofs will be given. The first has interest as it indicates the interaction between the $\mathbf{W}_i$'s, but it suffers from the defect that it cannot handle three-dimensional motion with a limiting collinear configuration. The goal of this first approach is to show that $W_1^2 = \langle\mathbf{W}_1, \mathbf{W}_1\rangle$ satisfies the differential equation.

$$\tfrac{1}{2}(W_1^2)' = -(\tfrac{1}{3} + o(1))\, W_1^2. \tag{2.3}$$

where $o(1)$ denotes terms which tend to zero as u approaches the appropriate limit.

If W_1^2 satisfies this equation, then W_1^2 must approach zero exponentially fast as $u \to +\infty$. This would complete the proof for completely parabolic motion. On the other hand, solutions of Eq. (2.3) are either identically zero or they approach infinitely exponentially fast as $u \to -\infty$. If the latter condition would occur for total collapse, it would violate the fact that $\mathbf{W}_1 \to \mathbf{0}$ as $u \to -\infty$. (Recall, $\mathbf{v} \to \mathbf{0}$.) Therefore, for total collapse problems, the former condition must hold and this more than satisfies our conclusion.

Equation (2.3) will be derived from Eq. (2.2) by taking the inner product of both sides of the second equation with respect to $\mathbf{W}_1$. Note that $\langle\mathbf{W}_1, \mathbf{W}_2\rangle = \langle\mathbf{W}_1, \mathbf{W}_3\rangle = 0$. Also, we claim that $\langle\mathbf{W}_1, (\tfrac{2}{9}\mathbf{R}_1 + m_1^{-1}\,\partial U/\partial\mathbf{R}_1, \dots, \tfrac{2}{9}\mathbf{R}_n + m_n^{-1}\,\partial U/\partial\mathbf{R}_n)\rangle = 0$. To see this, notice that the second vector is the gradient (with respect to this inner product) of J, and that J can be rewritten as $\sum_{1\leqslant i<j\leqslant n} m_i m_j(R_{ij}^2/9M + 1/R_{ij})$. (Here, we are using the center-of-mass constraint $\langle\mathbf{R}, \mathbf{E}_j\rangle = 0$.) Since the value of J depends only upon the mutual distances between configurations, it is rotation invariant. This in turn implies that the gradient of J is orthogonal to rotations, which proves our claim.

It follows from Eq. (2.2) and the above that

$$\langle\mathbf{W}_1', \mathbf{W}_1\rangle = -\tfrac{1}{3}\langle\mathbf{W}_1, \mathbf{W}_1\rangle - \langle\mathbf{W}_2', \mathbf{W}_1\rangle - \langle\mathbf{W}_3', \mathbf{W}_1\rangle. \tag{2.4}$$

To complete our derivation of Eq. (2.3), we must show $\langle\mathbf{W}_2', \mathbf{W}_1\rangle$, $\langle\mathbf{W}_3', \mathbf{W}_1\rangle = o(1)\, W_1^2$ as u approaches the appropriate sign of infinity.

Vector $\mathbf{W}_3 = \lambda\mathbf{R}$, so $\mathbf{W}_3' = \lambda'\mathbf{R} + \lambda(\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3)$, where λ is some scalar function of u. Therefore, $\langle\mathbf{W}_3', \mathbf{W}_1\rangle = \lambda\langle\mathbf{W}_1, \mathbf{W}_1\rangle$. Wintner showed for the total collapse problem that $I/t^{4/3}$ approaches a positive limit as $t \to 0$. Pollard [6] proved that completely parabolic motion exhibits the same behavior as $t \to \infty$. Therefore, in both types of motion, $\langle\mathbf{R}, \mathbf{R}\rangle$ approaches a positive limit as u approaches the appropriate limit. But since $\mathbf{v} \to \mathbf{0}$, this means that $\lambda \to 0$, or that $\langle \mathbf{W}_3', \mathbf{W}_1 \rangle = o(\langle \mathbf{W}_1, \mathbf{W}_1 \rangle)$.

Because $\langle \mathbf{W}_2, \mathbf{W}_1 \rangle = 0$, showing that $\langle \mathbf{W}_1', \mathbf{W}_2 \rangle = o(W_1^2)$ is equivalent to showing that $\langle \mathbf{W}_2', \mathbf{W}_1 \rangle = o(\mathbf{W}_1^2)$. We shall complete the proof of this theorem for all but motion in three-dimensional space tending to collinear central configurations by showing that the first condition holds. To do this, we use the fact that for each n and for each choice of the masses, there exists a positive constant bounding the distance from a non-collinear central configuration to a collinear configuration (Saari [11]). In effect, this means there exists a positive constant D such that if $\mathbf{A} \in \mathrm{CC}$ corresponds to a non-collinear central configuration, then for all $\mathbf{u} \in R^3$ of unit length, there is at least one index i such that $|\mathbf{u} \times \mathbf{A}_i| \geqslant D$, $\mathbf{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_n))$ and $\times$ is the vector product in R^3.

Let $\Omega^{-1}(u)$ correspond to the $SO(3)$ rotation of the configuration at time u. Then $\boldsymbol{\rho}(u) = \Omega(u)\,\mathbf{R}(u)$ corresponds to the change in the configuration with respect to this rotating coordinate system. Thus,

$$\boldsymbol{\rho}' = \Omega'\mathbf{R} + \Omega \boldsymbol{R}'$$

and

$$\begin{aligned} \mathbf{R}' &= -\Omega^{-1}\Omega'(\mathbf{R}) + \Omega^{-1}\boldsymbol{\rho}' \\ &= \mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3. \end{aligned}$$

Because $\Omega \in SO(3)$, $\Omega^{-1}\Omega'$ is a skew-symmetric; so there exists a vector $\mathbf{S}(u) \in R^3$ such that $-\Omega^{-1}\Omega'(\mathbf{R}) = (\mathbf{S} \times \mathbf{R}_1, \ldots, \mathbf{S} \times \mathbf{R}_n)$. This expression will be denoted by $\mathbf{S} \times \mathbf{R}$. Clearly, $\mathbf{W}_1 = \mathbf{S} \times \mathbf{R}$, so $\mathbf{W}_2 + \mathbf{W}_3 = \Omega^{-1}\boldsymbol{\rho}'$. If the motion is coplanar, then $\mathbf{S}$ is orthogonal to the plane of motion. In any case, $\mathbf{S}$ corresponds to the axis of rotation of the system in R^3.

Using this representation, we have from the fact $|\mathbf{R}|$ approaches a limit that there exists constant D_1 with the property that for sufficiently large value of $|u|$, $\langle \mathbf{W}_1, \mathbf{W}_1 \rangle = \langle \mathbf{S} \times \mathbf{R}, \mathbf{S} \times \mathbf{R} \rangle < D_1 S^2$. Furthermore, for coplanar motion and for motion where the defining central configurations are non-collinear, $\langle \mathbf{W}_1, \mathbf{W}_1 \rangle$ is bounded below by a positive multiple of S^2. This is so in the coplanar case because $\mathbf{S}$ is orthogonal to each R^3 component of $\mathbf{R}$; hence the magnitude of $\mathbf{S} \times \mathbf{R}_i$ is $|\mathbf{S}|\,|\mathbf{R}_i|$. The conclusion for the non-collinear configurations follows from our earlier statement bounding non-collinear central configurations from collinear configurations. We use this estimate to show that $\langle \mathbf{W}_1', \mathbf{W}_2 \rangle = o(S^2)$. (This will complete the proof.)

By direct computation,

$$\langle \mathbf{W}_1', \mathbf{W}_2 \rangle = \langle \mathbf{S}' \times \mathbf{R},\ \mathbf{W}_2 \rangle + \langle \mathbf{S} \times (\mathbf{W}_1 + \mathbf{W}_2 + \mathbf{W}_3), \mathbf{W}_2 \rangle.$$

Vector $\mathbf{S}' \times \mathbf{R}$ corresponds to a rotation of the configurations $\mathbf{R}$, so the first term on the right-hand side equals zero as $\mathbf{W}_2$ is orthogonal to the tangent space $T_x M_x$. All of the R^3 component vectors of $\mathbf{S} \times \mathbf{W}_2$ are orthogonal (with respect to usual R^3 inner product) to the corresponding components of $\mathbf{W}_2$. Thus $\langle \mathbf{S} \times \mathbf{W}_2, \mathbf{W}_2 \rangle = 0$. Since $\mathbf{W}_3$ is the radial velocity, $\langle \mathbf{S} \times \mathbf{W}_3, \mathbf{W}_2 \rangle = \langle \mathbf{S} \times \lambda \mathbf{R}, \mathbf{W}_2 \rangle = \lambda \langle \mathbf{W}_1, \mathbf{W}_2 \rangle = 0$.

Finally, $\langle \mathbf{S} \times \mathbf{W}_1, \mathbf{W}_2 \rangle = \langle \mathbf{S} \times (\mathbf{S} \times \mathbf{R}), \mathbf{W}_2 \rangle$. But, by the definitions of the R^3 and the $(R^3)^n$ inner products, and since the R^3 components of the $\mathbf{R}$ vector are bounded above, we have $|\langle \mathbf{S} \times (\mathbf{S} \times \mathbf{R}), \mathbf{W}_2 \rangle| \leqslant S^2 |\mathbf{W}_2|$. With the possible exception of motion in R^3 approaching a collinear central configuration, $S^2 |W_2|$ is bounded above by a positive multiple of $|\mathbf{W}_2| W_1^2$. Because $\mathbf{W}_2 \to \mathbf{0}$, it follows that $|\langle \mathbf{W}_1', \mathbf{W}_2 \rangle| = |\langle \mathbf{W}_2', \mathbf{W}_1 \rangle| = o(W_1^2)$, and this completes the proof of the theorem, except for the case of limiting collinear configurations.

The problem with motion in R^3 which approaches a collinear central configuration is that $\mathbf{S}(u)$ may approach the line in R^3 defined by the R^3 components of $\mathbf{R}$. In this case, W_1^2 need not serve as an upper bound for S^2. Therefore, without additional knowledge about the behavior of $\mathbf{W}_2$, the best we can say is that

$$\tfrac{1}{2} \langle \mathbf{W}_1, \mathbf{W}_1 \rangle' = -(\tfrac{1}{3} + o(1)) \langle \mathbf{W}_1, \mathbf{W}_1 \rangle + o(|\mathbf{S}|) \langle \mathbf{W}_1, \mathbf{W}_1 \rangle^{1/2}.$$

However, solutions of this equation need not decay fast enough to obtain the desired result.

3. Second Proof and Weierstrass's Theorem

The second approach yields a direct proof for the collision problem, but it involves an ad hoc construction for the proof of parabolic motion approaching a collinear configuration. Following this proof we extract, in a formal statement, the rotational nature of collapse orbits.

The scaling of variables converts the integral of angular momentum into the equation

$$\mathbf{c} e^{-1/3u} = \sum m_i \mathbf{R}_i \times \mathbf{R}_i' = \sum m_i \mathbf{R}_i \times (\mathbf{S} \times \mathbf{R}_i) + \Omega \left(\sum m_i \boldsymbol{\rho}_i \times \boldsymbol{\rho}_i' \right).$$

The second summation on the right-hand side equals zero because it involves components of $\mathbf{W}_2$ and $\mathbf{W}_3$ and they are velocity components without rotation.

If $\mathbf{S} \equiv \mathbf{0}$, there is no rotation, so the problem is completed. If $\mathbf{S} \not\equiv \mathbf{0}$, express $\mathbf{R}_i$ in terms of its component in the $\mathbf{S}$ direction and its component orthogonal to $\mathbf{S}$ as

$$\mathbf{R}_i = \lambda_i \mathbf{S}/|\mathbf{S}| + \boldsymbol{\eta}_i.$$

The integral of angular momentum is of the form

$$\mathbf{c}e^{-1/3u} = \mathbf{S} \sum m_i \eta_i^2 - S \sum m_i \lambda_i \boldsymbol{\eta}_i. \tag{3.1}$$

First we prove the theorem for total collapse. According to Weierstrass, if this occurs, then $\mathbf{c} = \mathbf{0}$. But, by construction, the two vectors given by the sums on the right-hand side of Eq. (3.1) are orthogonal; therefore they must both equal zero. In particular, this means that $\mathbf{S}$ and/or $\sum m_i \eta_i^2$ equals zero. But since $\langle \mathbf{W}_1, \mathbf{W}_1 \rangle = \langle \mathbf{S} \times \mathbf{R}, \mathbf{S} \times \mathbf{R} \rangle = S^2 \sum m_i \eta_i^2$, we have that $\mathbf{W}_1 \equiv \mathbf{0}$ for collapse problems. This completes the proof.

For completely parabolic motion confined to the algebraic variety $\mathbf{c} = \mathbf{0}$, the same conclusion that $\mathbf{W}_1 \equiv \mathbf{0}$ follows. So, assume that $\mathbf{c} \neq \mathbf{0}$. It now follows from Eq. (3.1) and the triangle inequality that

$$\left| \mathbf{S} \sum m_i \eta_i^2 \right| \leqslant ce^{-1/3u}. \tag{3.2}$$

If $\sum m_i \eta_i^2$ is bounded away from zero, then this inequality shows that $\mathbf{S}$ approaches zero exponentially fast; which in turn would complete the proof as this implies that $\mathbf{W}_1 = \mathbf{S} \times \mathbf{R}$ approaches zero exponentially fast. This is what happens if the limiting central configurations are not collinear. If the limiting configurations are collinear, $\sum m_i \eta_i^2$ still can be bounded away from zero as long as the angle between the axis of rotation, $\mathbf{S}/|\mathbf{S}|$, and the configuration is bounded away from zero; e.g., for coplanar motion where this angle is $\pi/2$.

To complete the proof, it seems easiest to compute the value of vector S. To do this, note that for any unit vector $\mathbf{l} \in R^3$, vector $\mathbf{l} \times \mathbf{R}$ corresponds to a rotation about the axis of rotation $\mathbf{l}$. The component of W_1 in this direction is given by $\langle \mathbf{R}', (\mathbf{l} \times \mathbf{R})/|\mathbf{l} \times \mathbf{R}| \rangle = |\mathbf{l} \times \mathbf{R}|^{-1} \sum m_i \mathbf{R}_i' \cdot (\mathbf{l} \times \mathbf{R}_i) = |\mathbf{l} \times \mathbf{R}|^{-1} \mathbf{l} \cdot (\sum m_i \mathbf{R}_i \times \mathbf{R}_i') = |\mathbf{l} \times \mathbf{R}|^{-1} \mathbf{l} \cdot \mathbf{c}e^{-u/3}$. Because the scalar product determines the component of $\mathbf{c}$ in the direction $\mathbf{l}$, it follows that if $\mathbf{c} = c\mathbf{k}$, then $\mathbf{W}_1 = |\mathbf{k} \times \mathbf{R}|^{-2} ce^{-u/3} \mathbf{k} \times \mathbf{R}$, $\mathbf{S} = ce^{-u/3} |\mathbf{k} \times \mathbf{R}|^{-2} \mathbf{k}$ and $|\mathbf{k} \times \mathbf{R}|^2 = \sum m_i \eta_i^2$.

If $\sum m_i \eta_i^2 \to 0$, then $\mathbf{R}$ approaches a limit which is the collinear central configuration along the z-axis. So, to complete the proof, we assume that $\limsup \sum m_i \eta_i^2 \geqslant A > 0$ and then show that this assumption is incompatible with $\liminf \sum m_i \eta_i^2 = 0$. To show this we will need information about $\mathbf{W}_2$; namely, how fast the particles approach their limiting configuration.

Suppose it is known that there is $\varepsilon > 0$ such that $|\mathbf{W}_2| < e^{-4\varepsilon u}$ for all sufficiently large values of u. If at time u_0, $|\mathbf{k} \times \mathbf{R}(u_0)|^2 = A > 0$, then $|\mathbf{k} \times \mathbf{R}(u)|^{-2} |\mathbf{W}_2|$ is exponentially small—at least until some future time after u_1 where $|\mathbf{k} \times \mathbf{R}(u_1)|^2 = Ae^{-2\varepsilon u_1}$. Thus in the time interval $[u_0, u_1]$, $S^2 |W_2|$ is bounded above by an exponentially small multiple of W_1^2. This means that should u_0 be chose large enough; then Eq. (2.3) would be satisfied where the $o(1)$ term is bounded below by $-\varepsilon/2$. Thus on this time interval, $W_1^2(u) \leqslant W_1^2(u_0)\, e^{-2(u-u_0)/3}\, e^{\varepsilon(u-u_0)}$. From the equation for $\mathbf{W}_1$ we obtain the inequality $|\mathbf{k} \times \mathbf{R}(u)|^2 \geqslant Ae^{-\varepsilon(u-u_0)}$. According to the definition of u_1, this implies that $u_1 = \infty$. Thus Eq. (2.3) is satisfied for all $u \geqslant u_0$, and $\mathbf{W}_1$ approaches zero exponentially fast.

All that remains is to show that $\mathbf{W} = \mathbf{W}_2 + \mathbf{W}_3$ decreases exponentially fast for these orbits. But, substituting the expression for $\mathbf{W}_1$ into the second of Eq. (2.2), performing the indicated differentiation on $\mathbf{W}_1$, collecting like terms and using the fact that $W_1 \to 0$, it follows that $\mathbf{W}' = -\frac{1}{3}\mathbf{W} + \nabla J + o(W)$. This means that the solution will have exponential decay if all the eigenvalues of the right-hand side evaluated at the appropriate central configuration have non-zero real parts. It will be established in the next section that this is true for all collinear central configurations. This completes the proof.

We conclude this section by reemphasizing the following statement concerning total collapse which improves upon Weierstrass' result.

THEOREM 3.1. *Systems admitting a total collapse have no natural rotational motion. Namely,* $W_1 \equiv 0$ *for all time the solution exists.*

4. STABLE MANIFOLD

In this final section we obtain a further refinement of Theorem 2.1 by showing that for most values of the masses in the coplanar problem and for an open set of masses in general the stable and the unstable sets of the points of CC are smooth submanifolds where the dimensions of the submanifolds depend upon whether it is a stable or unstable set and upon the type of central configuration being approached. This conclusion turns out to be a consequence of an analysis of the behavior of $\mathbf{W}_2 + \mathbf{W}_3$. Combined with the earlier analysis of $\mathbf{W}_1$, this provides a fairly complete description of collapse and completely parabolic motion.

A central configuration corresponds to a critical point of IU^2 (Wintner [13, p. 273]). Since IU^2: $R^{3(n-1)} \to R$ is rotation invariant and homogeneous of degree zero, the Hessian of IU^2 evaluated at central configuration $\mathbf{A}$ must contain in its kernel tangent vectors corresponding to these directions. That is, the kernel contains the product of $T_{\mathbf{A}}M_{\mathbf{A}}$ and the subspace $\{\mathbf{V} \mid \mathbf{V}$ is a scalar multiple of $\mathbf{A}\}$. If this product describes the kernel of the Hessian, then central configuration $\mathbf{A}$ will be called non-degenerate.

Although the definitions differ, this definition for non-degeneracy is equivalent to the one used by Palmore in his study of coplanar central configurations. Palmore showed for the coplanar central configurations that for fixed n and for "most" choices of the masses, the central configurations are non-degenerate. The same is true in the general case for at least an open set of masses. Of course, this means that these central configurations are isolated. (More precisely, the equivalence class as determined by rotations and scalar change of variable are isolated.) This fact was used in the previous section to show there are systems not admitting a continuum of central configurations. We use these statements and the stable manifold theorem to prove the much stronger result that the stable and unstable sets of these configurations are smooth submanifolds.

THEOREM 4.1. *For a non-degenerate central configuration, the stable and the unstable sets of this point are smooth submanifolds. The dimension of the stable manifold is greater than or equal to the dimension of the unstable manifold.*

The actual dimensions of the manifolds are determined by the properties of the central configuration. Some examples are given in what follows. Of course, once the structure of the stable and the unstable set of **A** are determined, we can use symmetry with respect to $SO(3)$ to determine that the stable and unstable sets of $M_{\mathbf{A}}$ are also manifolds where the dimensions are augmented by the $SO(3)$ action. This is the effect of the following statement, which does this for all choices of **A** admitted as central configurations in a given n-body problem.

THEOREM 4.2. *For those choices of n and those choices of masses where all central configurations are non-degenerate, the set of initial conditions leading to total collapse and the set of initial conditions leading to completely parabolic motion are the finite union of smooth submanifolds. Each submanifold is of lower dimension.*

Proof of Theorem 4.1. Let $\mathbf{A} \in \mathrm{CC}$ correspond to a non-degenerate central configuration. About **A**, Eq. (2.2) become

$$\begin{pmatrix} \mathbf{R} \\ \mathbf{V} \end{pmatrix}' = \begin{pmatrix} 0 & I \\ B & (-1/3)I \end{pmatrix} \begin{pmatrix} \mathbf{R} \\ \mathbf{V} \end{pmatrix} + \text{h.o.t.},$$

where $B = D(\nabla J)_{\mathbf{A}}$, I is the identity matrix of the appropriate dimension, and h.o.t. means "higher-order terms." To use the stable manifold theorem, we need to determine the eigenvalues of the linear term, that is,

$$\begin{pmatrix} 0 & I \\ B & (-1/3)I \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

This leads to the equations $\mathbf{v} = \lambda \mathbf{u}$ and $B\mathbf{u} = \frac{1}{3}\mathbf{v} + \lambda \mathbf{v}$, or $B\mathbf{u} = (\frac{1}{3}\lambda + \lambda^2)\mathbf{u}$. Consequently, if μ is an eigenvalue and $\mathbf{u}$ is the corresponding eigenvector for B, then $(\mathbf{u}, \lambda\mathbf{u})$ is an eigenvector and λ is an eigenvalue for our original system. The values of λ are given by the equation $\lambda^2 + \frac{1}{3}\lambda - \mu = 0$. This means that each non-zero eigenvalue for B gives two non-zero eigenvalues for the original system. We will show that μ must be real-valued. Therefore, it follows from the DeCartes rule of signs that a positive value of μ yields one positive and one negative value for λ, a negative value for μ yield two values for λ, both with negative real part, while if $\mu = 0$, then $\lambda = -\frac{1}{3}, 0$. From this it follows that the number of eigenvalues with negative real part is greater than or equal to the number of positive eigenvalues. This will prove our assertion concerning the dimension of the submanifolds.

To determine the signs of the eigenvalues of B by use of the assumption that $\mathbf{A}$ is non-degenerate, we express J in terms of IU^2. According to their definitions

$$\nabla J = \left(\frac{2}{9}\mathbf{R}_1 + m_1^{-1}\frac{\partial U}{\partial \mathbf{R}_1}, \ldots, \frac{2}{9}\mathbf{R}_n + m_n^{-1}\frac{\partial U}{\partial \mathbf{R}_n}\right)$$

and

$$\nabla IU^2(\mathbf{R}) = \left(U^2(\mathbf{R})\mathbf{R}_1 + m_1^{-1} 2IU(\mathbf{R})\frac{\partial U}{\partial \mathbf{R}_1}, \ldots, U^2(\mathbf{R})\mathbf{R}_n + m_n^{-1} 2IU(\mathbf{R})\frac{\partial U}{\partial \mathbf{R}_n}\right).$$

Since $\mathbf{A}$ is a central configuration with scale factor 2/9, it follows from the second equation that $(U/2I)(\mathbf{A}) = 2/9$. Therefore we have

$$\nabla J(\mathbf{A}) = f(\mathbf{A})\,\nabla IU^2(\mathbf{A}) \qquad \text{where} \quad f(\mathbf{R}) = (2IU(\mathbf{R}))^{-1}.$$

The above expression holds only at $\mathbf{R} = \mathbf{A}$. However, using the fact $\nabla I(\mathbf{R}) = \mathbf{R}$, then following relationship holds everywhere:

$$\nabla J(\mathbf{R}) = f(\mathbf{R})\,\nabla IU^2(\mathbf{R}) + g(\mathbf{R})\,\nabla I \qquad \text{where} \quad g(\mathbf{R}) = \left(\frac{2}{9} - \frac{U}{2I}(\mathbf{R})\right).$$

Therefore by identifying the various gradients with mappings from $R^{3(n-1)}$ to R^{3n}, we have

$$\begin{aligned} B = D(\nabla J(\mathbf{R}))|_{\mathbf{A}} &= D(f(\mathbf{R})\,\nabla IU^2(\mathbf{R}) + g(\mathbf{R})\,\nabla I)|_A \\ &= f(\mathbf{A})\,D(\nabla IU^2)|_{\mathbf{A}} + g(\mathbf{A})\,D(\nabla I)_A + IU^2(\mathbf{A})\,Df(\mathbf{A}) + \mathbf{A}D\mathbf{g}_A. \end{aligned}$$

Since $g(\mathbf{A}) = 0$ and $\nabla IU^2(\mathbf{A}) = \mathbf{0}$, only the first and the last terms on the right-hand side remain. ($\mathbf{A}$ is a column vector and $Dg_{\mathbf{A}}$ is represented by a row vector.)

We now show that $\mathbf{A}$ is an eigenvector of B. This is because $B\mathbf{A} = f(\mathbf{A})\,D(\nabla IU^2)_{\mathbf{A}}\mathbf{A} + (\mathbf{A}Dg_A)\mathbf{A}$. But vector $\mathbf{A}$ in $T_{\mathbf{A}}R^{3(n-1)}$ corresponds to scalar change in the configuration. Since IU^2 is homogenerous of degree zero, $\mathbf{A}$ is in the kernel of $D(\nabla IU^2)_{\mathbf{A}}$; so that expression becomes $B\mathbf{A} = (\mathbf{A}Dg_A)\mathbf{A} = (Dg_a(\mathbf{A}))\mathbf{A}$. That is, $\mathbf{A}$ is an eigenvector with eigenvalue $Dg_A(\mathbf{A})$.

We now show that the eigenvalue $Dg_A(\mathbf{A}) = 2/9$. According to its definition, $2Dg_{\mathbf{A}} = -(I(\mathbf{A}))^{-1} DU_{\mathbf{A}} + (U/I^2)(\mathbf{A})\, DI$. Because $\mathbf{A}$ is a central configuration, $DU_{\mathbf{A}} = \frac{2}{9}(m_1\mathbf{A}_1, ..., m_n\mathbf{A}_n)$ and $DI_{\mathbf{A}} = (m_1\mathbf{A}_1, ..., m_n\mathbf{A}_n)$. Therefore

$$2Dg_{\mathbf{A}}(\mathbf{A}) = \left(\frac{-2}{9I(\mathbf{A})} + \frac{U}{I^2}(\mathbf{A})\right) \langle \mathbf{A}, \mathbf{A} \rangle = \frac{2}{9I(\mathbf{A})} \langle \mathbf{A}, \mathbf{A} \rangle = \frac{4}{9},$$

and the conclusion follows. The corresponding values for λ are $\frac{1}{3}$ and $-\frac{1}{3}$ and $-\frac{2}{3}$. (Notice, these are the eigenvalues for motion corresponding to $\mathbf{W}_3$.)

Matrix $\mathbf{A}Dg_A$ has rank one. Furthermore, a direct computation shows that any vector orthogonal to $\mathbf{A}$ lies in the kernel of this matrix. On the other hand, by the homogenity of IU^2, any eigenvector of $D(\nabla IU^2)$ is orthogonal to $\mathbf{A}$. Thus, the remaining eigenvectors and eigenvalues of B are those of $f(\mathbf{A})\, D(\nabla IU^2)_A$.

By assumption $\mathbf{A}$ is non-degenerate. Therefore, the kernel of B is three dimensional, and it corresponds to $T_{\mathbf{A}}M_{\mathbf{A}}$. (This kernel is the tangent space of the rotational motion which was discussed in Sections 2 and 3.) The corresponding eigenvalues for λ are 0 and $-1/3$, both with multiplicity three.

We now show that our assumption $\mathbf{A}$ is non-degenerate implies that the remaining eigenvalues are real and non-zero. To do this, let M be the diagonal matrix with entry m_i in rows $3i-2$, $3i-1$, and $3i$. Then

$$D(\nabla IU^2) = M^{-1} D^2 IU^2.$$

D^2IU^2 is a symmetric matrix, so its eigenvalues are all real. By our assumption that $\mathbf{A}$ is non-degenerate, the eigenvalues are all non-zero (except, of course, for those eigendirections corresponding to scalar change or rotation.) Since M^{-1} is positive definite and D^2IU^2 is hermitian, the eigenvalues of $M^{-1}D^2IU^2 = D(\nabla IU^2)$ also are real and they have the same inertias; that is, they have the same number of positive, negative, and zero eigenvalues. (This is classical theorem. A recent generalization can be found in Johnson [3].)

This implies that the remaining eigenvalues for B are all real and non-zero. Furthermore, because of the M^{-1} matrix, even if the eigenvalues of D^2IU^2 do not depend upon the masses, those of B do. The signs of the eigenvalues depend upon the nature of the central configuration (see Saari [10]). For example, the equilateral triangle for $n = 3$, and the equilateral tetrahedron for $n = 4$ correspond to a minimum of IU^2. The collinear configurations are local minimum of IU^2 in directions along the line, but local maxima for directions in higher dimensions. Indeed, this holds in general: the eigenvalues corresponding to a change from a coplanar configuration to a three-dimensional configuration are all negative. We shall use this fact in what follows.

Since the only eigenvalues with zero real part correspond to $\mathbf{W}_1$ motion, motion which does not exist for the collapse problem and which approaches limit in complete parabolic motion, we can use the stable manifold theorem to reach the desired conclusion. The dimension of the stable manifold of a point is determined by the number of eigenvalues with negative parts, while the dimension of the unstable manifold corresponds to the number of positive eigenvalues. This completes the proof of the theorem.

THEOREM 4.3. *If an orbit terminates in a total collapse where the limiting configuration is collinear, then the motion was confined to a straight line for all time.*

Proof. As pointed out above, a collinear central configuration $\mathbf{A}$ corresponds to saddle point of IU^2 whęre the only positive eigenvalues are for eigenvectors corresponding to collinear motion. Therefore the unstable manifold of $\mathbf{A}$ when viewed as a problem in $(R^3)^n$ is of the same dimension as when viewed as a problem in $(R)^n$, which is physical space for rectilinear motion. Consequently they agree, and the unstable manifold of a collinear configuration in CC corresponds to rectilinear motion. This statement is *not* true for parabolic motion because the dimensions of the stable manifold change with the dimension of physical space.

THEOREM 4.4. *If a total collapse orbit terminates in a coplanar central configuration, then the motion is coplanar for all time it exists.*

Proof. The proof is similar to that given for Theorem 4.3. It uses the fact that the eigenvalues of B corresponding to non-coplanar directions are all negative. Hence they lead to eigenvalues of the full system with negative real parts. As such the unstable manifold for the two-dimensional problem must agree with that for the three-dimensional problem.

THEOREM 4.5. *In general, total collapse corresponds to an essential singularity.*

Proof. Since some of the eigenvalues depend continuously upon the value of the masses, there are values of the masses giving rise to eigenvalues, λ, with irrational values, and by use of the stable manifold theorem, this motion approaches zero like $e^{\lambda u}$. Converting back to the time variable t, this means that a series expansion about zero will includes terms t_{λ}^{-1}, terms with irrational exponents. The means that if the singularity is isolated, it is an essential singularity.

A corresponding statement holds for completely parabolic orbits when the independent variable is expanded about the point ∞; that is, when expanded about the north pole on the Riemann sphere.

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